

ONE CAN HEAR THE SHAPE OF A TRIANGLE

DANIEL GRIESER AND SVENJA MARONNA

ABSTRACT. In 1966 Mark Kac asked the famous question 'Can one hear the shape of a drum?'. While this was later shown to be false in general, it is known that one can hear the shape of a triangle. We will give a new proof of this fact. The central point of the argument is to show that area, perimeter and the sum of the reciprocals of the angles determine a triangle uniquely. This is proved using convexity arguments and the partial fraction expansion of $\sin^{-2} x$.

1. INTRODUCTION

In 1966 Mark Kac asked the famous question 'Can one hear the shape of a drum?' in [7]. Mathematically, the question was whether the eigenvalues of the Dirichlet Laplacian on a bounded planar domain determine the domain uniquely (these terms are explained below). After many efforts the question was finally answered, in the negative, in 1992 by C. Gordon, D. Webb and S. Wolpert [6]. In the case of triangles, however, C. Durso showed in 1990 [5] that the answer is affirmative: The Dirichlet eigenvalues determine the shape of a triangle uniquely. We give a new proof of this fact which is interesting in two respects: First, the input from the theory of partial differential equations which is used is much simpler than in Durso's proof. All we need here was already known in the 1960s and will be recalled below; and second, the central step in the proof is to establish the following rather peculiar geometric fact about triangles, which is interesting in its own right.

Main Theorem. *A triangle is determined uniquely by its area, its perimeter and the sum of the reciprocals of its angles.*

To put this into perspective, let us recall a few facts from high school. There we learn that a triangle is determined by three pieces: The three side lengths, or two side lengths and the enclosed angle, or one side length and two angles. This may be stated as saying that the space of triangles is three-dimensional. We also learn that not *any* three pieces determine a triangle uniquely. For example, fixing two side lengths and an angle not enclosed by them usually leaves two choices for the triangle. Or prescribing the three angles only determines the triangle up to dilation, that is, up to a continuum of choices. For the latter the reason is easy to find – the three angles are not 'independent', they always sum to π , so instead of three we might as well have prescribed only two of the angles. There are different ways to make the notion of independence precise – for example, algebraically or differentially –, and without going into further detail here it may be stated that three independent quantities will usually determine a point in a three-dimensional space – so in our

2010 *Mathematics Subject Classification.* 35P99, 35R30, 51N20 .

Key words and phrases. Inverse spectral problem, heat trace asymptotics, convexity, partial fraction expansion.

case a triangle – up to finitely many choices. Now the three quantities in our Main Theorem really look like they are independent (and it follows from our proof that they are). So the point of the theorem is that, given these three quantities, there is not just a finite number but precisely one triangle.

So why on earth *reciprocals of angles*? Has anyone ever heard of such a quantity? We certainly haven't – at least not in the realm of geometry. We now explain where this quantity arises, and why the Main Theorem implies the claim in the title. If you prefer you may jump ahead to the proof of the theorem in Section 2. In Section 3 you find a different perspective on the proof and more remarks about the Dirichlet eigenvalues.

The relation to hearing the shape of a triangle. Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with piecewise smooth boundary $\partial\Omega$, for example a triangle. The Laplacian of a function u on Ω is the function $\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$. The eigenvalue problem for the Laplacian on Ω , with Dirichlet boundary conditions, is to find solutions u, λ to the equations

$$\begin{aligned} \Delta u &= -\lambda u && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Here $\lambda \in \mathbb{R}$ and u is a continuous function on the closure $\bar{\Omega}$ which is twice differentiable on Ω . The number λ is called a Dirichlet eigenvalue of Ω if there is a solution u which is not the constant zero. The sign is chosen so that any eigenvalue is positive. One can prove (see [4]) that the set of eigenvalues forms a sequence $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots \rightarrow \infty$. For each eigenvalue, the set of solutions u is a finite dimensional vector space, and the sequence of eigenvalues is written so that an eigenvalue appears as often as the dimension of this vector space indicates.

The eigenvalues have the following physical meaning. Think of Ω as a drum, i.e. a membrane which is stretched over a wire frame in the shape of $\partial\Omega$. The membrane can vibrate freely except that it is fixed at the boundary. When the drum vibrates you will hear a sound, which is composed of tones of various frequencies. These frequencies are the numbers $\gamma\sqrt{\lambda_k}$, where γ is a constant depending on the material and tension of the drum. So if we suppose that γ is known to the listener then in this sense we can 'hear' the eigenvalues λ_k .

If Ω was one-dimensional (an interval) instead of 2-dimensional then its Dirichlet eigenvalues (with Δu replaced by u'') would be $\lambda_k = Ck^2$ where $C = \frac{\pi^2}{L^2}$, with L the length of the interval, so the vibration frequencies would be $\frac{\gamma\pi}{L}k$, that is, integer multiples of the base frequency $\frac{\gamma\pi}{L}$. These are what in music are called the harmonics: tones which sound simultaneously with the base note when a string is plucked, bowed or struck, or when another essentially one-dimensional body is vibrating, for example the column of air in a wind instrument.

It should be remarked that this is an idealized physical model, for real drums (or strings) the frequencies are slightly different due to non-linear effects and the influence of the resonance chamber. Also note that, unlike for strings, the 'harmonics' (then called overtones) generated by drums are not integer multiples of the base (lowest) frequency, which explains why drums often do not have a clear pitch. However, instrument builders have invented various ways to change at least a few of the overtones to be harmonic, i.e. integer multiples of the base frequency, for

example by using suitable resonance chambers or by coating part of the membrane with a paste (as for the Indian tabla).

Back to mathematics: We are in the following situation: To each domain Ω we have associated a sequence of numbers $\lambda_1, \lambda_2, \dots$. This begs for mathematical investigation! Can we calculate the λ_k ? No, except in very few cases. Can we say anything interesting how they depend on the shape of Ω ? Yes. This is the subject of the mathematical discipline called spectral geometry (see [2] for a short introduction and more references, and also [4] and [8]). Can we recover the domain Ω from knowing the eigenvalues? This is the famous *inverse problem* formulated at the beginning of this article. One of the fundamental tools in studying the inverse problem is the function

$$(1) \quad h(t) = \sum_{k=1}^{\infty} e^{-\lambda_k t}, \quad t > 0$$

which is called the trace of the heat kernel because of its relation to the way heat flows in the domain Ω . By investigating the solutions of the heat equation one can show that the series defining h converges for $t > 0$ and that $h(t)$ has an asymptotic expansion as $t \rightarrow 0$, which in the case of polygons Ω reads as follows:

$$h(t) = a_0 t^{-1} + a_1 t^{-\frac{1}{2}} + a_2 + O(e^{-\frac{c}{t}}) \quad \text{as } t \rightarrow 0$$

for some constant $c > 0$, where

$$a_0 = \frac{A}{4\pi}, \quad a_1 = \frac{P}{8\sqrt{\pi}}, \quad a_2 = \frac{1}{24} \sum_i \left(\frac{\pi}{\alpha_i} - \frac{\alpha_i}{\pi} \right)$$

where A is the area, P the perimeter and the α_i are the interior angles of the polygon. This formula was first mentioned in [9], the first published proof was given in [10]. In the case of the triangle we have $\sum_i \alpha_i = \pi$, so $a_2 = \frac{\pi}{24} \sum_{i=1}^3 \frac{1}{\alpha_i} - \frac{1}{24}$.

Therefore, if we know all the λ_k then we know the function $h(t)$ and hence the coefficients a_0, a_1, a_2 , hence the area, the perimeter and the sum of the reciprocals of the angles of the triangle. So we can hear these quantities, and then by the Main Theorem we can hear the shape of the triangle.

Durso used in her proof the wave kernel $w(t) = \sum_{k=1}^{\infty} e^{i\sqrt{\lambda_k} t}$, which is related to the propagation of waves in Ω . This sum only converges in the sense of distributions, and by analyzing solutions of the wave equation (which is technically more intricate than for the heat equation) she shows that the distribution w has a smallest positive singularity t_0 , which equals the length of the shortest altitude in the case of an obtuse or right-angled triangle, and the perimeter of the triangle formed by the base points of the three altitudes in the case of an acute triangle. Area and perimeter can also be read off from w , and these three quantities again determine the triangle uniquely.

2. PROOF OF THE MAIN THEOREM

We denote the angles of the triangle by α, β, γ , its area by A and its perimeter by P . We use the following formula from triangle geometry¹:

$$(2) \quad \frac{P^2}{4A} = \cot \frac{\alpha}{2} + \cot \frac{\beta}{2} + \cot \frac{\gamma}{2}$$

This allows us to work exclusively with angles. We will prove:

Proposition. *A triple (α, β, γ) of positive real numbers satisfying $\alpha + \beta + \gamma = \pi$ is uniquely determined, up to ordering, by the values of*

$$(3) \quad f(\alpha, \beta, \gamma) = \cot \frac{\alpha}{2} + \cot \frac{\beta}{2} + \cot \frac{\gamma}{2}$$

$$(4) \quad g(\alpha, \beta, \gamma) = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}.$$

The Main Theorem follows directly from this: if the area A and perimeter P are given then the angles are determined by equation (2) and the Proposition, so the triangle is determined up to dilation. Then the given area fixes the dilation factor.

So it remains to prove the Proposition. One way to proceed would be to eliminate one of the variables, say α , using the relation $\alpha = \pi - \beta - \gamma$, then eliminate another variable (say β) from the given value of g by solving a quadratic equation, then plug the expressions for α and β into f and investigate the resulting equation for γ . But this is horrible! Even if it works, it is ugly mathematics. If nothing else, the beautiful symmetry present in the statement of the Proposition is lost.

Symmetry is a treasure. One should keep it and use it as long as possible. This is what we shall do.

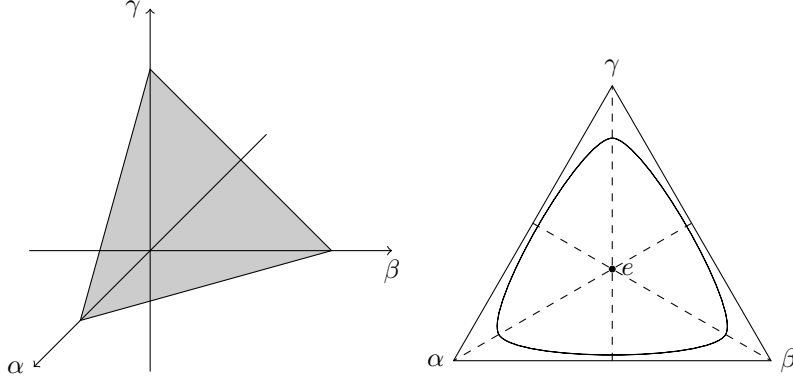
Proof of the Proposition. Let $D = \{(\alpha, \beta, \gamma) : \alpha, \beta, \gamma > 0, \alpha + \beta + \gamma = \pi\} \subset \mathbb{R}_{>0}^3$ where $\mathbb{R}_{>0} = (0, \infty)$. We think of points of D as 'marked triangles up to dilation', where 'marked' means that we have named the angles in a certain order. The set D is (the interior of) a triangle itself – the triangle cut out of the plane $\alpha + \beta + \gamma = \pi$ by the positive octant, see Figure 1. Points on the dashed lines correspond to isosceles triangles, the center e corresponds to the equilateral triangle. Let us call a point which does not lie on a dashed line a *non-isosceles point*. The non-isosceles points form six connected subsets, which we call *chambers*. The dashed lines are also lines of symmetry: If we pick a non-isosceles point and reflect it step by step across all dashed lines, we obtain six points, one in each chamber; these six points correspond to the same triangle, with angles named in different orders. Each chamber corresponds to one ordering of the angles, for example the lower left chamber to the ordering $\alpha > \beta > \gamma$, or $\alpha \geq \beta \geq \gamma$ when we include its dashed boundary parts.

The idea of the proof is to show that the level sets of the function g are convex curves, see Figure 1, and that f is strictly monotone along the part of any one of these curves lying in one chamber.

Lemma 1.

a) *The function g is strictly convex on $\mathbb{R}_{>0}^3$.*

¹We are grateful to Richard Laugesen for pointing out this identity. Amazingly, both sides are also equal to the product $\cot \frac{\alpha}{2} \cot \frac{\beta}{2} \cot \frac{\gamma}{2}$; it's a nice little exercise in addition theorems to prove this.

FIGURE 1. The space of angles of a triangle, and a level line of g

- b) *The gradients $\nabla f, \nabla g, \nabla h$ are linearly independent at all non-isosceles points of D .*

Let us finish the proof of the Proposition and then return to prove Lemma 1. The strict convexity of g implies that the sublevel set $G_{\leq t} = \{p \in \mathbb{R}_{>0}^3 : g(p) \leq t\}$ is strictly convex for any $t > 0$, with boundary the level surface $G_t = \{p \in \mathbb{R}_{>0}^3 : g(p) = t\}$. Furthermore, these sets are symmetric under all permutations of the coordinates. These properties then also hold for the intersections of the sublevel and level sets with the plane $\alpha + \beta + \gamma = \pi$. Since $g(p) \rightarrow \infty$ when p approaches the boundary of D (i.e. when at least one of the angles tends to zero), it follows that the sets $G_t \cap D$ are either closed curves in the interior of D which encircle the point e , or the point e , or empty. Since the equilateral triangle has $g(e) = \frac{9}{\pi}$, the first case corresponds to $t > \frac{9}{\pi}$.

In particular, we see that the point e is already determined by the value of g alone².

Now consider any level curve $G_t \cap D$ with $t > \frac{9}{\pi}$. Consider the arc of the curve running inside one chamber, with endpoints p, q corresponding to isosceles triangles. Our proof will be complete if we can show that f is strictly monotone along this part of the curve.

Suppose f was not strictly monotone. Then there would be a point r on this arc, different from p and q , where f is stationary, that is, its derivative along the arc vanishes. By the Lagrange multiplier theorem this would mean that $\nabla f(r)$ is a linear combination of $\nabla g(r)$ and $\nabla h(r)$. But this would be a contradiction to part b) of the Lemma. This completes the proof of the Proposition. \square

²This can also be seen from the arithmetic-harmonic mean inequality $3 \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} \right)^{-1} \leq \frac{\alpha + \beta + \gamma}{3}$ with equality iff $\alpha = \beta = \gamma$. Yet another proof of our Main Theorem for the case of the equilateral triangle follows from the isoperimetric theorem for triangles: Among all triangles with given perimeter, the equilateral triangle has the largest area, and it is unique with this area. This implies that for this case already area and perimeter suffice to determine the triangle. It can be proved using the formula $A = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{P}{2}$ and a, b, c are the side lengths, using the arithmetic-geometric mean inequality.

Proof of Lemma 1. a) The Hessian (matrix of second derivatives) of g is the diagonal matrix with entries $\frac{2}{\alpha^3}, \frac{2}{\beta^3}, \frac{2}{\gamma^3}$ on the diagonal. This is clearly positive definite for all $(\alpha, \beta, \gamma) \in \mathbb{R}_{>0}^3$, and this implies that g is strictly convex.

b): We have

$$\nabla f = -\frac{1}{2} \begin{pmatrix} \frac{1}{\sin^2 \frac{\alpha}{2}} \\ \frac{1}{\sin^2 \frac{\beta}{2}} \\ \frac{1}{\sin^2 \frac{\gamma}{2}} \end{pmatrix}, \quad \nabla g = - \begin{pmatrix} \frac{1}{\alpha^2} \\ \frac{1}{\beta^2} \\ \frac{1}{\gamma^2} \end{pmatrix}, \quad \nabla h = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Suppose there is a non-isosceles point (α, β, γ) (i.e. these numbers are pairwise different) and numbers R, S, T , not all zero, with $R\nabla f + S\nabla g + T\nabla h = 0$. This would mean that the function

$$F(y) = -\frac{R}{2} \frac{1}{\sin^2 \frac{y}{2}} - S \frac{1}{y^2} + T$$

had three different zeroes in the interval $(0, \pi)$, namely $y = \alpha$, $y = \beta$ and $y = \gamma$. In order to show that this cannot happen we prove that the function F is a non-zero constant, or strictly monotone, or strictly concave or convex on this interval, depending on the values R, S, T . Below we will prove:

Lemma 2. *The function $G(x) = \frac{1}{\sin^2 x} - \frac{1}{x^2}$ is strictly increasing and strictly convex on the interval $(0, \pi)$.*

This lemma implies that the function $G_C(x) = \frac{1}{\sin^2 x} - \frac{C}{x^2}$ is, on the interval $(0, \pi)$, strictly increasing for $C \geq 1$ and strictly convex for $C \leq 1$, since $G_C(x) = G(x) + \frac{1-C}{x^2}$ and the function $\frac{1-C}{x^2}$ is increasing for $C > 1$ and convex for $C < 1$. Now clearly for any values of R, S, T we can write $F(y)$ as a constant multiple of $G_C(\frac{y}{2})$, for some C , plus a constant, and the claim follows. \square

Proof of Lemma 2. First note that this is non-trivial: It is easy to check that both $\frac{1}{\sin^2 x}$ and $\frac{1}{x^2}$ have positive second derivative whenever they are defined, hence are convex, but it is not clear why their difference should be convex. However, things become very transparent when we use the series representation (partial fraction expansion)

$$\frac{1}{\sin^2 x} = \sum_{k=-\infty}^{\infty} \frac{1}{(x - k\pi)^2}$$

which follows from the well-known partial fraction expansion of the cotangent by differentiation. This yields $G(x) = \sum_{k \neq 0} \frac{1}{(x - k\pi)^2}$. Now every summand $\frac{1}{(x - k\pi)^2}$ is

strictly convex on $(0, \pi)$ since the function $\frac{1}{x^2}$ is strictly convex on both half lines $x < 0$ and $x > 0$, so G is strictly convex. Furthermore, the series shows that G is regular at $x = 0$, and it is also even, so $G'(0) = 0$. Combined with strict convexity this implies that G is strictly increasing on the interval $(0, \pi)$, which was to be shown. \square

3. FURTHER REMARKS

Let us take another look at the proof, from a slightly different perspective. The Proposition, which implies the Main Theorem by elementary triangle formulas, may be restated as saying that the map $\Phi = (f, g) : D \rightarrow \mathbb{R}^2$ is injective on the closure

in D of each chamber. The proof of injectivity has two ingredients: First, Lemma 1 b), which may be restated as saying that the differential of the map Φ is invertible in the chamber and hence, by the inverse mapping theorem, that Φ is locally injective everywhere, that is, every point of the chamber has a neighborhood on which Φ is injective. Second, the convexity of Lemma 1 a) allows to infer global injectivity from this local statement. Finally, the analytic core of the whole argument is Lemma 2 which is used in the proof of Lemma 1 b). We now take another look at this.

A different proof of Lemma 2. While the trick using the partial fraction representation is very elegant, you might wonder if there is a more pedestrian way to prove convexity of G . Indeed there is. Here is a sketch (it was our first proof of this result): A short calculation gives $\frac{1}{2}G''(x) = \frac{3}{\sin^4 x} - \frac{2}{\sin^2 x} - \frac{3}{x^4}$. We need to show that this is positive (here and in the sequel we always assume $x > 0$). This is equivalent to the inequality

$$(5) \quad 3 \sin^4 x + 2x^4 \sin^2 x \stackrel{!}{<} 3x^4$$

How can one prove an inequality involving trigonometric functions and polynomials? Maybe your first idea is to use the well-known inequality $\sin x < x$ to get rid of the sines. But clearly this does not help since $3x^4 + 2x^4 \cdot x^2 > 3x^4$. How can we do better?

Recall where the inequality $\sin x < x$ comes from: x is the first term in the Taylor series of $\sin x$, the next term is negative. Of course this is not a proof, but it's the core idea, which can be turned into a proof as follows: The function $f(x) = x - \sin x$ vanishes at $x = 0$ and has derivative $f'(x) = 1 - \cos x$, which is always non-negative, and is positive for small positive x . Thus, $x - \sin x > 0$ for all positive x follows by integration: $f(x) = \int_0^x f'(t) dt > 0$.

So in order to prove (5) we can try to use a better estimate for $\sin x$ by using more terms from its Taylor series. We have the estimate

$$(6) \quad \sin x < x - \frac{x^3}{6} + \frac{x^5}{120}$$

This can be proved in the same way as $\sin x < x$: The function $f(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \sin x$ satisfies $f(0) = f'(0) = f''(0) = f'''(0) = f^{(4)}(0) = 0$ and $f^{(5)}(x) = 1 - \cos x \geq 0$, and > 0 for small positive x . So integrating we obtain $f^{(4)}(x) = \int_0^x f^{(5)}(t) dt > 0$, then integrating again we get $f'''(x) > 0$ and so forth, until we obtain $f(x) > 0$ for all $x > 0$.³

We now plug (6) into the left hand side of (5). A rather tedious calculation shows that the result, which starts as $3x^4 - \frac{1}{15}x^8 + \dots$, is less than $3x^4$ for $x < 4$.

³Instead we could have used Taylor's formula with remainder for the function $g(x) = \sin x$:

$$g(x) = \sum_{k=0}^4 \frac{x^k}{k!} + \frac{1}{4!} \int_0^t (x-t)^4 g^{(5)}(t) dt$$

which using $g^{(5)}(t) = \cos t \leq 1$ (and < 1 for small positive t) and $\int_0^t (x-t)^4 dt = \frac{1}{5}x^5$ yields the same result. Yet another proof uses Leibniz' criterion for the Taylor series $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$ of $\sin x$, which is alternating. The terms after the fifth power are monotonically decreasing in absolute value if $\frac{x^{2n+1}}{(2n+1)!} < \frac{x^{2n-1}}{(2n-1)!}$ for $n \geq 4$, which is equivalent to $x^2 < 2n(2n+1)$, hence true for $x < \sqrt{72}$. Since the first omitted term after $\frac{x^5}{5!}$ is negative, we get that the sum of the series, which is $\sin x$, is less than $x - \frac{x^3}{3!} + \frac{x^5}{5!}$, at least for $x < \sqrt{72}$. Since $\sqrt{72} > \pi$, this is enough for our purpose.

The main point is that the second term is negative. Among the higher terms some are positive, but they can easily be estimated against the negative ones.

More on the inverse spectral problem for triangles. The way in which the Dirichlet eigenvalues determine the triangle is somewhat indirect: First one constructs the heat kernel h , see (1), and then considers the coefficients in its asymptotic expansion to prove the result. In particular, one needs to know *all* the eigenvalues for this. It is natural to ask whether already a finite number of eigenvalues, ideally only three, suffice to determine the triangle. Indeed, numerical evidence was provided in [1] that $\lambda_1, \lambda_2, \lambda_3$ determine a triangle uniquely – but $\lambda_1, \lambda_2, \lambda_4$ do not. However, no proof of this is known. As a partial result in this direction it is proved in [3] that for each $\varepsilon > 0$ there is a number N so that $\lambda_1, \dots, \lambda_N$ determine a triangle uniquely among all triangles whose angles are all greater than or equal to ε .

Let us end this article with another unsolved problem.

Problem. *Can one hear the shape of a convex polygon?*

We emphasize that the answer is no when convexity is not required: The counterexamples to ‘Can one hear the shape of drum?’ in [6] are non-convex octagons.

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INSTITUT FÜR MATHEMATIK, CARL VON OSSIETZKY UNIVERSITÄT OLDENBURG, 26111 OLDENBURG, GERMANY

E-mail address: daniel.grieser@uni-oldenburg.de

INSTITUT FÜR MATHEMATIK, CARL VON OSSIETZKY UNIVERSITÄT OLDENBURG, 26111 OLDENBURG, GERMANY

E-mail address: svenja.maronna@uni-oldenburg.de